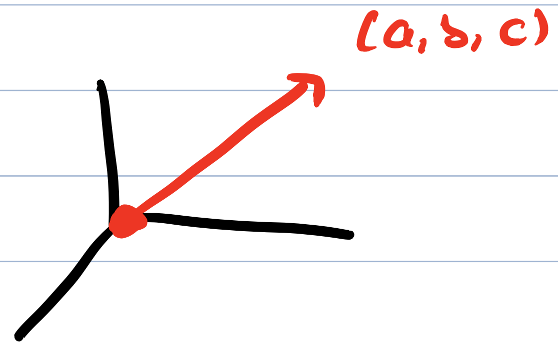
Calculus III.

Vectors



$\vec{v} = (a, b, c)$

magnitude / length : $|\vec{v}| = \sqrt{a^2+b^2+c^2}$

direction : $\dfrac{\vec{v}}{|\vec{v}|}$

$$\vec{v} = \text{magnitude} \cdot \text{direction}$$
$$= |\vec{v}| \cdot \frac{\vec{v}}{|\vec{v}|}$$

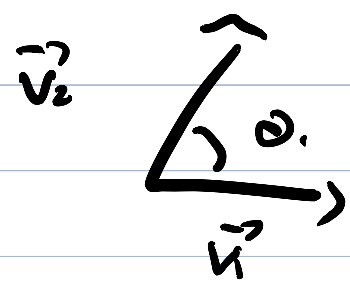
ijk- notation : $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$

$i = (1, 0, 0)$ $\quad j = (0, 1, 0)$ $\quad K = (0, 0, 1)$

$\vec{v_1} = (a_1, b_1, c_1)$ $\quad \vec{v_2} = (a_2, b_2, c_2)$

dot product : $\vec{v_1} \cdot \vec{v_2} = a_1 a_2 + b_1 b_2 + c_1 c_2$

geometric interpretation :



$$\vec{v_1} \cdot \vec{v_2} = |\vec{v_1}|\,|\vec{v_2}|\cos\theta$$

$$\cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}|\,|\vec{v_2}|}$$

$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0.$

perpendicular

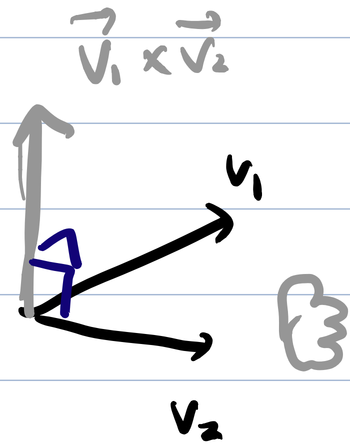
<u>**Cross product**</u>:

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= (b_1 c_2 - b_2 c_1)\, i + (a_2 c_1 - a_1 c_2)\, j + (a_1 b_2 - a_2 b_1)\, k.$$

geometric interpretation:

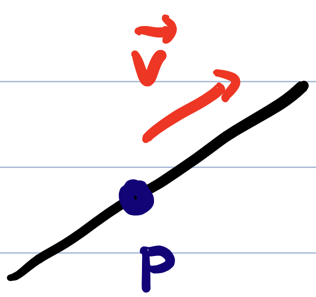
$$|\vec{v_1} \times \vec{v_2}| = |\vec{v_1}|\, |\vec{v_2}| \sin\theta.$$



right-hand rule.

<u>**Lines**</u>

A line is determined by $\left[\begin{array}{ll} \text{a point} & P \\ \text{direction} & \vec{v} \end{array}\right.$



$$\boxed{L: \ \vec{r}(t) = P + t \cdot \vec{v}}$$

vector equation

Let $P = (x_0, y_0, z_0)$ $\vec{v} = (v_1, v_2, v_3)$

$$\Rightarrow \boxed{\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}}$$ parametric equation

$$t = \boxed{\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}}$$ symmetric equation

e.g Find a parametric equation of the line passing through $P = (-3, -1, 2)$ and $Q = (5, 8, 4)$

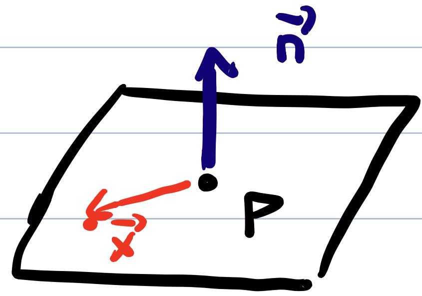
Sol
$$\left[\begin{array}{l} \text{point : } P = (-3, -1, 2) \\ \text{direction : } \vec{v} = Q - P = (8, 9, 2) \end{array} \right.$$

$$\therefore \quad \begin{aligned} x &= -3 + 8t \\ y &= -1 + 9t \\ z &= 2 + 2t \end{aligned} \qquad \square$$

Planes

A plane in 3-space is determined by

- a point : P
- normal vector : $\vec{n}$



$\vec{x} = (x, y, z)$

$\Rightarrow \boxed{(\vec{x} - P) \cdot \vec{n} = 0}$

Let $\vec{P} = (x_0, y_0, z_0)$ $\quad \vec{n} = (a, b, c)$

$\Rightarrow \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

e.g. Find the equation of the plane containing

$(1,0,0) \quad (0,1,0) \quad (0,0,1)$

$P = (1, 0, 0)$

$\vec{v}_1 = (0,1,0) - (1,0,0) = (-1, 1, 0)$

$\vec{v}_2 = (0,0,1) - (1,0,0) = (-1, 0, 1)$

$\vec{n} = \vec{v}_1 \times \vec{v}_2 = (1, 1, 1)$

$(\vec{x} - P) \cdot \vec{n} = 0 \Rightarrow (x - 1) + y + z = 0$

$$\Rightarrow \quad x + y + z = 1 .$$

Partial derivatives

$f : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notation: $\frac{\partial f}{\partial x}$, f_x, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, ...

Prop $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose f has continuous second partial derivatives. Then

$$f_{xy} = f_{yx}$$

Prop (Chain rule) $f : \mathbb{R}^2 \to \mathbb{R}$, $x, y : \mathbb{R} \to \mathbb{R}$. Suppose f has continuous partial derivatives and x, y have continuous derivatives. Then

$$\frac{d}{dt}(f(x(t), y(t))) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt}(t) +$$

$$\frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}(t).$$

<u>Gradient</u> $\nabla f(x,y) := (f_x(x,y), f_y(x,y))$

$\nabla f(x,y,z) := (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$

<u>geometric meaning</u> : normal vector of a level surface $f(x,y,z) = c$

<u>e.g</u> Tangent plane of a surface $z = f(x,y)$

$\rightarrow (x, y, f(x,y))$ at (x_0, y_0)

Let $g(x,y,z) = z - f(x,y)$

The surface $z = f(x,y)$: level surface of g at 0.

$\left[\begin{array}{ll} \text{point:} & P = (x_0, y_0, f(x_0, y_0)) \\ \text{normal:} & \nabla g(x_0, y_0, z_0) \quad (z_0 = f(x_0, y_0)) \end{array}\right.$

$= (-f_x(x_0, y_0), -f_y(x_0, y_0), 1)$

$(\vec{x} - P) \cdot \vec{n} = 0$

$\Rightarrow \quad -f_x \cdot (x - x_0) - f_y \cdot (y - y_0) + (z - z_0) = 0$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

<u>geometric meaning 2.</u> $\nabla f(x, y)$ is the direction in which f increases most rapidly

$-\nabla f(x, y)$ is the direction in which f decreases most rapidly

<u>Directional derivatives</u> $\vec{u}$: unit vector $(|\vec{u}| = 1)$

$f: \mathbb{R}^2 \to \mathbb{R}$ differentiable

$$D_u f(x, y) = \nabla f \cdot \vec{u}$$